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Propagation of a Pulse in an Inhomogeneous Medium

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Abstract

This report deals with the propagation of a pulse in a semi-infinite inhomogeneous medium for which the law of variation of the refractive index is such that the wave equation can be solved by the method of separation of variables. The Laplace transform of the solution is first expressed as a series of eigenfunctions, whose asymptotic behavior at distant points of the Laplace-plane is investigated by means of improved variants of the W.K.B. method. An approximation is found whose inverse Laplace transform can be calculated; it applies when a shadow, in the usual sense of geometrical optics, is formed, and it describes the variation of the solution immediately behind the diffracted front which propagates into the shadow. The work is an extension of the author's recent report on diffraction of pulses by a circular cylinder [3].

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Introduction

In a recent report[3] the author discussed the diffraction of two-dimensional pulses by a circular cylinder. The object of the present report is to extend this discussion to a more general class of propagation and scattering problems.

Let ϕ be a function of two space coordinates x and y and of the time t, which satisfies the two-dimensional wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \quad ,$$

where c, the velocity of wave propagation, may be a function of x and y. This is the equation satisfied by the pressure (or its time integral) of an acoustic disturbance in an atmosphere whose temperature is not necessarily uniform, or, if c is constant, by the transverse components of TE - or TM-waves. One may consider the propagation of waves governed by this equation in the space exterior to a cylinder with generators parallel to the z-axis, on whose surface either ϕ or its normal derivative is to vanish. But it is always possible to effect a conformal transformation of x and y which maps this 'physical space' into the upper half-plane, $y \geq 0$, while preserving both the form of the wave equation and of the boundary condition. It is therefore sufficient to consider the propagation problem in an inhomogeneous, semi-infinite medium with either of the boundary conditions $\phi = 0$ or $\partial\phi/\partial y = 0$ at $y = 0$. The basic problem is the determination of the Green's function, G, which can be defined as the disturbance generated by an instantaneous point source. Using Dirac's δ-function, one can say that the Green's function satisfies the equation

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial y^2} = \delta(x - x_o)\delta(y - y_o)\delta(t), \qquad (y_o > 0)$$

and also the initial conditions

$$G = 0, \quad \frac{\partial G}{\partial t} = 0, \quad \text{when} \quad t = 0, \quad (x - x_o)^2 + (y - y_o)^2 \neq 0 ,$$

with the appropriate boundary condition at $y = 0$. G is the disturbance caused by an instantaneous source at $x = x_o$, $y = y_o$ at the time $t = 0$; the disturbance due to such a source acting at time t_o is of course simply $G(x,y,t - t_o,x_o,y_o)$.

If the medium were of infinite extent, the Green's function would reduce to Hadamard's elementary solution divided by 2π. Let $\rho = \rho(x,y;x_o,y_o)$ be the time, as determined by Fermat's principle, when a wave starting at $t = 0$ from (x_o,y_o) arrives at the point (x,y). Then the elementary solution, G_o, vanishes for $t < \rho$, and is given at least for small positive $t - \rho$ by a series of the form

$$G_o = \sum_{n=0}^{\infty} E_n(x,y)\,(t^2 - \rho^2)^{n-\frac{1}{2}} \quad .$$

The coefficient E_n can be determined recursively, essentially by integration of the wave equation along the ray from (x_o,y_o) to (x,y).

If there is a boundary at $y = 0$ then a ray from (x_o,y_o) meets $y = 0$ at, say, $(x_1,0)$, and one can construct a ray reflected from $(x_1,0)$. If (x,y) is a point on this reflected ray, then one can determine the time required for a wave to travel from (x_o,y_o) to $(x_1,0)$ along the direct ray, and from $(x_1,0)$ to (x,y) along the reflected ray. This total time is a function of position, and will be denoted by ρ^*. Now it is possible to construct another solution, G^*, of the wave equation. This solution vanishes for $t < \rho^*$, and for $t > \rho^*$ it is of the form

$$G^* = \sum_{n=0}^{\infty} E_n^* (x,y) \, (t^2 - \rho^{*2})^{n-\frac{1}{2}} \quad ;$$

further, G^* is such that on $y = 0$ either $G^* = - G_o$ or $\partial G^*/\partial y = - \partial G_o/\partial y$. The E_n^* are obtained recursively by integration of the wave equation along a reflected ray. Clearly,

$$G = G_o + G^*$$

satisfies the conditions of the problem. The leading coefficients, E_o and E_o^*, can be obtained easily; the approximation which results when only the leading terms are retained corresponds to the results obtained by the methods of geometrical optics.

However, it may happen that there are points of the xy-plane which cannot be joined to (x_o, y_o) by means of a direct and a reflected ray. All the domains whose points have this property are called the shadow, and in this case we shall say there is diffraction. In the shadow neither ρ^* nor the E_n^* can be defined, and there arises the problem of finding an alternative representation of G.

The present report gives a partial solution of this problem. It is confined to the case where the refractive index is of the form

$$\frac{1}{c^2} = f(x) + g(y) \, ,$$

where the functions $f(x)$, $g(y)$ are subject to certain restrictions, the most important of which is that $g(y)$ must be increasing and tend to infinity with y. An example discussed in Appendix II suggests furthermore that the solution obtained may not be valid in the whole shadow.

The method used is similar to that used in the previous report [3]. First, the Laplace transform $\bar{\phi}$ of ϕ, namely

$$\bar{\phi} = \int_0^\infty e^{-st}\, \phi \, dt ,$$

is expanded as a series of eigenfunctions which represent modes propagated parallel to the boundary (Section 3). Next, the asymptotic behavior of the eigenvalues as $Re(s) \to \infty$ is investigated. This is done by a variant of Langer's method (Sections 4 - 6), which was used in the cylinder diffraction problem. [An alternative method has recently been found and is described briefly in Appendix I. This method is simpler and - at least from a formal point of view - more effective.] It is then possible to derive approximations valid for $Re(s)$ large, for the terms of the series for $\bar{\phi}$. The form of these approximations suggests that they are valid in the shadow only, and shows that only the leading term of the first mode may be retained (Sections 7 and 8). In Section 9 the results obtained are related to the geometrical optics of the problem. The final step, namely the calculation of the inverse Laplace transform of the approximation obtained for $\bar{\phi}$, is effected by appealing to results already obtained in the case of diffraction by a circular cylinder. Since that case is typical of the case of diffraction problems we are considering, the present report repeats the methods and calculations used in the earlier work; however, the results obtained are more general and can be used, for instance, to calculate the diffracted pulse in the shadow of an elliptic or a parabolic cylinder (see Section 2).

1. Formulation of the problem

The problem can be formulated analytically as follows. It is required to find a function ϕ of two variables x,y (the space coordinates) and the time t which satisfies, in the domain $y \geq 0$, the wave equation

$$(1.1) \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left[f(x) + g(y)\right] \frac{\partial^2 \phi}{\partial t^2} - \delta(x - x_o)\delta(y - y_o)\,\delta(t), \quad y \geq 0$$

and the initial conditions

$$(1.2) \qquad \phi = \frac{\partial \phi}{\partial t} = 0, \quad t = 0,\ y > 0,\ (x - x_o)^2 + (y - y_o)^2 \neq 0 .$$

In addition, either of the boundary conditions

$$(1.3) \qquad \left[\phi\right]_{y=0} = 0$$

or

$$(1.4) \qquad \left[\frac{\partial \phi}{\partial y}\right]_{y=0} = 0$$

is to hold at $y = 0$. The symbol δ in (1.1) stands for the Dirac δ-function. The functions f(x) and g(y) are assumed to satisfy such conditions of regularity (or analyticity) as the argument may require, as well as the following conditions:

$$(1.5) \qquad g(0) > 0 ;$$

$$(1.6) \qquad f(x) \geq 0 ;$$

$$(1.7) \qquad g'(y) > 0 \quad \text{for} \quad y > 0 ; \quad g(y) \to \infty \ \text{as} \ y \to \infty .$$

2. Illustrative examples

As already mentioned in the Introduction, problems of the type just described arise in the discussion of the diffraction of two-dimensional pulses. To illustrate this, let us consider the following situation for

definiteness. Suppose that ϕ is the velocity potential of an acoustic disturbance. Consider first a homogeneous medium in which the space coordinates are (x', y'); then we have

$$\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} , \tag{2.1}$$

where c is a constant. We may then consider the field due to a pulse represented by Hadamard's elementary solution

$$\phi_o = \frac{1}{2\pi} \frac{H(ct - R)}{(c^2 t^2 - R)^{1/2}} , \quad R^2 = (x' - x_o')^2 + (y' - y_o')^2 \tag{2.2}$$

(H denotes Heaviside's unit function). This field may be derived by including a δ-function 'source' term in (2.1) analogous to that in (1.1), and adding initial conditions analogous to (1.2). Let the reflector (scattering object) be bounded by a surface S at which either $\phi = 0$ or $\partial\phi/\partial n = 0$, where $\partial/\partial n$ denotes differentiation along the normal. It may happen that a conformal mapping of the exterior of S onto the upper half of the xy-plane transforms (2.1) into an equation of the type (1.1); then the scattering problem is equivalent to that formulated in Section 1.

Example I. Circular cylinder. The conformal mapping

$$x' + iy' = e^{-i(x + iy)}$$

transforms the exterior of the unit circle $x'^2 + y'^2 = 1$ into $y \geq 0$, and (2.1) into (1.1) with $f(x) = 0$, $g(y) = c^{-2} e^{2y}$. In this case, ϕ must be a periodic function of x with period 2π. But it can be expressed as the sum of terms, each of which satisfies the conditions of the problem of Section 1.

Example II. Parabolic cylinder. The conformal mapping

$$x' + iy' = \frac{1}{2}(x + iy)^2$$

transforms the exterior of the parabola $y'^2 > 4(x' + 1)$, into the region $y \geq \sqrt{2}$. At the same time, (2.1) becomes (1.1) with $f(x) = x^2/c^2$, $g(y) = y^2/c^2$. The reduction of the scattering problem to that of Section 1 - by a translation of the origin along the y-axis and a change of scale - is trivial.

Example III. Diffraction by an elliptic cylinder. Let the mapping be

$$x' + iy' = \sin(x + iy) \quad .$$

Then $f(x) = c^{-2} \cos 2x$ and $g(y) = c^{-2} \cosh 2y$, and $y \geq a$ corresponds to

$$\frac{x'^2}{\cosh^2 a} + \frac{y'^2}{\sinh^2 a} \geq 1 \quad .$$

This again gives essentially the problem of Section 1, with the same additional difficulty that occurs in the case of Example I (circular cylinder); this can be circumvented in the same manner.

3. The formal solution

To obtain a formal solution of the problem stated in Section 1, we begin by introducing the Laplace transform $\bar{\phi}$ of ϕ, i.e.,

$$\bar{\phi}(x,y,s) = \int_0^\infty e^{-st}\ \phi(x,y,t)\,dt\ . \tag{3.1}$$

This satisfies the equation (see [3] p. 6, footnote)

$$\frac{\partial^2 \bar{\phi}}{dx^2} + \frac{\partial^2 \bar{\phi}}{dy^2} = s^2\left[f(x) + g(y)\right]\bar{\phi} - \delta(x-x_o)\,\delta(y-y_o) \tag{3.2}$$

and either of the boundary conditions $\bar{\phi} = 0$ or $\partial\bar{\phi}/\partial y = 0$ at $y = 0$. In addition, as a consequence of the general properties of Laplace transforms, $\bar{\phi} \to 0$ as $\mathrm{Re}(s) \to \infty$. (In practice this appears to be equivalent to the 'radiation condition' $\bar{\phi} \to 0$ as $y \to \infty$). Since this last condition may be taken to be a consequence of the assumption that $\phi = 0$ for $t < 0$, we shall refer to it as the 'pulse condition'.

We can now obtain a formal solution of (3.2) as an eigenfunction expansion by using the method of normal coordinates of the theory of vibrations. Consider the solutions of the ordinary differential equation

$$\frac{d^2\chi}{dy^2} + \left(\lambda - s^2\, g(y)\right)\chi = 0 \tag{3.3}$$

where λ is a real or complex parameter. For sufficiently large y, the coefficient of χ is negative, and there are two independent solutions of (3.3) which, for large s, are asymptotic to

$$\left[s^2 g(y) - \lambda\right]^{-1/4} \exp\left\{\pm \int^y (s^2 g(y) - \lambda)^{1/2}\, dy\right\}\ .$$

If we require $\chi \to 0$ as $\mathrm{Re}(s) \to \infty$, then we must choose the minus sign, and the solution thus determined has the property that it tends to zero as $y \to \infty$. Let $\chi(y,s,\lambda)$ denote such a solution of (3.3). Since $g(y) \to \infty$ as $y \to \infty$, this solution is determined except for a factor independent of y. Each of the boundary conditions $\chi = 0$ at $y = 0$ or $\chi' = 0$ at $y = 0$ is satisfied only for a sequence of values of λ, say $\lambda_1, \lambda_2 \ldots$, the eigenvalues of the problem and the corresponding solutions $\chi(y,s,\lambda)$ form a complete orthogonal set ([7] Ch. VII).

Also, if $\lambda \neq \lambda_n$, we have both

$$\int_0^\infty \chi(y,s,\lambda)\, \chi''(y,s,\lambda_n)\,dy = \int_0^\infty \chi(y,s,\lambda)\chi(y,s,\lambda_n)\left[s^2 g(y) - \lambda_n\right]dy$$

and

$$\int_0^\infty \chi(y,s,\lambda)\chi''(y,s,\lambda_n)\,dy = \left[\chi(y,s,\lambda)\chi_n'(y,s,\lambda_n) - \chi'(y,s,\lambda)\chi(y,s,\lambda_n)\right]_0^\infty$$

$$+ \int_0^\infty \chi''(y,s,\lambda)\chi(y,s,\lambda_n)\,dy$$

$$= \chi'(0,s,\lambda)\chi(0,s,\lambda_n) - \chi(0,s,\lambda)\chi'(0,s,\lambda_n)$$

$$+ \int_0^\infty \chi(y,s,\lambda)\chi(y,s,\lambda_n)\left[s^2 g(y) - \lambda\right]dy .$$

Hence

$$(3.4)\qquad \int_0^\infty \chi(y,s,\lambda)\chi(y,s,\lambda_n)\,dy = \frac{\chi'(0,s,\lambda)\chi(0,s,\lambda_n) - \chi(0,s,\lambda)\chi'(0,s,\lambda_n)}{\lambda - \lambda_n}$$

and so, if we let $\lambda \to \lambda_n$, we obtain by d'Hôpital's rule

$$(3.5)\qquad \int_0^\infty \left[\chi(y,s,\lambda_n)\right]^2 dy = \left[\frac{\partial \chi'(0,s,\lambda)}{\partial\lambda}\,\chi(0,s,\lambda_n) - \frac{\partial\chi(0,s,\lambda)}{\partial\lambda}\,\chi'(0,s,\lambda_n\right]_{\lambda=}$$

where, because of the boundary conditions, one of the terms on the right-hand side is zero. If we denote the right-hand side of (3.5) by ω_n then a complete normalized orthogonal set is formed by the functions

$$(3.6)\qquad \chi_n(y,s) = \frac{\chi(y,s,\lambda_n)}{\omega_n^{1/2}} \quad .$$

Now suppose that

$$(3.7)\qquad \bar{\phi}(x,y,s) = \sum_{n=1}^{\infty} \Psi_n(x,s)\,\chi_n(y,s) \quad .$$

Then

$$(3.8)\qquad \Psi_n(x,s) = \int_0^\infty \bar{\phi}(x,y,s)\,\chi_n(y,s)\,dy \quad .$$

Multiply (3.2) by χ_n and integrate with respect to y from 0 to ∞. Then, since

$$\chi_n'' = \left(s^2 g(y) - \lambda_n\right)\chi_n$$

and since both χ_n and $\bar{\phi}$ satisfy the boundary conditions at y = 0, it follows (if we assume that $\bar{\phi} \to 0$ as $y \to \infty$) that

$$(3.9)\qquad \frac{d^2\Psi_n}{dx^2} - \left(s^2 f(x) + \lambda_n\right)\Psi_n = -\chi_n(y_o)\,\delta(x - x_o) \quad .$$

Thus, at $x = x_o$,

$$(3.10)\qquad \left[\Psi_n\right]_{x_o - 0}^{x_o + 0} = 0\,, \qquad \left[\frac{d\Psi_n}{dx}\right]_{x_o - 0}^{x_o + 0} = -\chi_n(y_o) \quad .$$

When s is large, the homogeneous equation corresponding to (3.9) has two solutions which are asymptotic to

$$K(s^2 f(x) + \lambda)^{-1/2} \exp\left\{ \pm \int_o^x \left[s^2 f(x) + \lambda_n \right]^{1/2} dx \right\} ,$$

where K is some constant. Let us denote these by Ψ_n^+, Ψ_n^- respectively. We then have

$$\Psi_n = \frac{\Psi_n^+(x,s)\,\Psi_n^-(x_o,s)}{W_n}\,\chi_n(y_o) \qquad (x \geq x_o)$$

(3.11)

$$= \frac{\Psi_n^+(x_o,s)\,\Psi_n^-(x,s)}{W_n}\,\chi_n(y_o) \qquad (x \leq x_o) ,$$

where $W_n = W_n(s)$ is the Wronskian

$$W_n(s) = \Psi_n^+ \frac{d\Psi_n^-}{dx} - \Psi_n^- \frac{d\Psi_n^+}{dx} . \tag{3.12}$$

Finally, by (3.8) and (3.6),

$$\bar{\phi}(x,y,s) = \sum_{n=1}^{\infty} \frac{\Psi_n^+(x,s)\,\Psi_n^-(x_o,s)\,\chi(y,s,\lambda_n)\,\chi(y_o,s,\lambda_n)}{W_n(s)\,\omega_n(s)} \tag{3.13}$$

for $x > x_o$. If $x < x_o$, then x and x_o must be interchanged.

4. Asymptotic approximations for Re(s) large

Just as in the case of diffraction by a circular cylinder, the essential step now is the derivation of approximation formulas for (3.13) that are asymptotically valid as $\mathrm{Re}(s) \to \infty$. For this purpose, the W.K.B. method could be used, except that the dependence of any λ_n (n fixed) on s is such that the error in the W.K.B. approximation does not tend to zero as $s \to \infty$.

A more elaborate approximation to $\chi(y,s,\lambda)$ is therefore required, and can be obtained by following Langer's procedure [5]. For simplicity, we shall only discuss the case of real positive s.

Let

$$\lambda = s^2 \mu \ . \tag{4.1}$$

If can be shown that the eigenvalues $\lambda_1, \lambda_2, \ldots$, are in the range $\mu \geq g(0)$. Hence we take $\mu \geq g(0)$. Then $g(y) - \mu$ changes sign when $y = y^* = y^*(\mu)$, where $g(y^*) = \mu$, and we can define a real function η of y by the equations

$$\begin{aligned} \frac{2}{3}\,\eta^{3/2} &= \int_{y^*}^{y} \left[g(y') - \mu\right]^{1/2} dy', && y \geq y^*; \\ \frac{2}{3}\,(-\eta)^{3/2} &= \int_{y}^{y^*} \left[\mu - g(y')\right]^{1/2} dy', && y \leq y^* \ . \end{aligned} \tag{4.2}$$

Then η has the same sign as $y - y^*$, and

$$\frac{d\eta}{dy} = \left(\frac{g(y) - \mu}{\eta}\right)^{1/2} \ . \tag{4.3}$$

It is easily shown that $d\eta/dy > 0$, and that η and its derivatives up to any required order are finite at $y = y^*$, if g(y) is sufficiently regular. The behavior of η as $y \to \infty$ depends on that of y; for instance, if $g \sim y^m$ for large y then

$$\eta \sim y^{(2m+2)/3} \quad \text{and} \quad d\eta/dy \sim y^{(m-1)/3} \ .$$

Let

$$X(y) = \left(\frac{\eta}{g(y) - \mu}\right)^{1/4} \mathrm{Ai}(s^{2/3}\eta) = \left(\frac{d\eta}{dy}\right)^{-1/2} \mathrm{Ai}(s^{2/3} y) \ , \tag{4.4}$$

where the Airy functions $\mathrm{Ai}(\gamma)$ satisfies the differential equation

(4.5) $$Ai''(\gamma) = \gamma\, Ai(\gamma)$$

and is given by (see [6])

(4.6) $$Ai(\gamma) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{z^3}{3} + \gamma z\right) dz = \frac{\gamma^{1/2}}{\sqrt{3}\,\pi}\, K_{1/3}\left(\frac{2\gamma^{3/2}}{3}\right)$$

As a second solution of (4.5) we take

(4.7) $$Bi(\gamma) = \frac{2\,\gamma^{1/2}}{\sqrt{3}\,\pi}\, I_{1/3}\left(\frac{2\,\gamma^{3/2}}{3}\right) ,$$

so that

(4.8) $$Ai(\gamma)\, Bi'(\gamma) - Ai(\gamma)\, Bi(\gamma) = \frac{1}{\pi} .$$

Now

$$\frac{dX}{dy} = s^{2/3}\left(\frac{d\eta}{dy}\right)^{1/2} Ai(s^{2/3} y) + \left[\frac{d}{dy}\left(\frac{d\eta}{dy}\right)^{-1/2}\right] Ai(s^{2/3} y) ,$$

and so

$$\frac{d^2X}{dy^2} = s^{4/3}\left(\frac{d\eta}{dy}\right)^{3/2} Ai''(s^{2/3}\eta) + Ai(s^{2/3}\eta)\, \frac{d^2}{dy^2}\left(\frac{d\eta}{dy}\right)^{-1/2} .$$

Using (4.3), (4.4) and (4.5) we can write this as

(4.9) $$\frac{d^2X}{dy^2} = \left\{ s^2\,[g(y) - \mu] + q(y) \right\} X$$

$$q(y) = \left[\frac{d\eta}{dy}\right]^{1/2} \frac{d^2}{dy^2} \left[\frac{d\eta}{dy}\right]^{-1/2} .$$

As the second solution of (4.9) we define

(4.10) $$Y = \left[\frac{d\eta}{dy}\right]^{-1/2} Bi\,(s^{2/3}\eta) .$$

It then follows from (4.8) that

$$X\,Y' - X'\,Y = \frac{s^{2/3}}{\pi} \quad . \tag{4.11}$$

5. An integral equation for χ

We can now derive an integral equation satisfied by $\chi(y)$. In the first place, since both $X \to 0$, $\chi \to 0$ as $y \to \infty$, we have from (4.9) and (3.3)

$$\int_y^\infty \chi''\,X\,dy = s^2 \int_y^\infty (g-\mu)\,\chi\,X\,dy \quad ,$$

$$\int_y^\infty \chi''\,X\,dy = \left[\chi'X - \chi\,X'\right]_y^\infty + \int_y^\infty \left[s^2(g-\mu) + q\right]\chi\,X\,dy \quad .$$

Hence

$$\chi'(y)\,X(y) - \chi(y)\,X'(y) = \int_y^\infty X(y_1)\,q(y_1)\,\chi(y_1)\,dy_1 \quad .$$

Similarly,

$$\chi'(y)Y(y) - \chi(y)Y'(y) = \chi'(0)Y(0) - \chi(0)Y'(0) - \int_0^y Y(y_1)q(y_1)\chi(y_1)dy_1 \quad .$$

Since χ can be multiplied by an arbitrary constant, we may suppose that

$$\chi'(0)\,Y(0) - \chi(0)\,Y'(0) = \frac{s^{2/3}}{\pi} \quad .$$

If then follows at once that, upon eliminating $\chi'(y)$ from the preceding equations, we have

$$\chi(y) = X(y) + \frac{\pi}{s^{2/3}}\left\{\int_0^y X(y)Y(y_1)q(y_1)\chi(y_1)dy_1 + \int_y^\infty X(y_1)Y(y)q(y_1)\chi(y_1)dy_1\right\} . \tag{5.1}$$

If we put

$$(5.2)\qquad \chi^*(y) = \chi(y)\left[\frac{d\eta}{dy}\right]^{1/2}, \quad q^*(y) = \left[\frac{d\eta}{dy}\right]^{-1/2}\left\{\frac{d^2}{dy^2}\left[\frac{d\eta}{dy}\right]^{-1/2}\right\}$$

then (5.1) assumes the form

$$(5.3)\qquad \chi^*(y) = Ai(s^{2/3}\eta) + \frac{\pi}{s^{2/3}}\left\{\int_0^y Ai(s^{2/3}\eta)Bi(s^{2/3}\eta_1)q^*(y_1)\chi^*(y_1)dy_1 + \int_0^\infty Ai(s^{2/3}\eta_1)\,Bi(s^{2/3}\eta)\,q^*(y_1)\chi^*(y_1)\,dy_1\right\},$$

where η_1 is related to y_1 as η is related to y.

We now make the assumption that both the integrals

$$\int_0^\infty |q^*(y)|\,dy, \qquad \int_0^\infty |q^*|^2\,\frac{d\eta}{dy}\,dy$$

converge uniformly in μ in any bounded μ-interval, and we put

$$(5.3a)\qquad K = K(\mu) = \int_0^\infty |q^*(y)|\,dy\ .$$

Then, since $|\chi(y)|^2$ is integrable (according to a fundamental theorem of Weyl), we have, by Schwarz' inequality,

$$\int_0^\infty |q^*\chi^*|dy \le \left\{\int_0^\infty \chi^2 dy\right\}^{1/2}\left\{\int_0^\infty |q^*|^2\,\frac{d\eta}{dy}\,dy\right\}^{1/2} < \infty\ .$$

Hence an upper bound for the integral terms in (5.3) is

$$\frac{\pi}{s^{2/3}} \left\{ \left| Ai(s^{2/3}\eta) \right| \max_{0 \le y_1 \le y} \left| Bi(s^{2/3}\eta_1) \right| \int_0^y |q^*\chi^*| dy_1 \right.$$

(5.4)

$$\left. + \left| Bi(s^{2/3}\eta) \right| \max_{y_1 \ge y} \left| Ai(s^{2/3}\eta_1) \right| \int_y^\infty |q^*\chi^*| dy_1 \right\} .$$

Now for $\eta > 0$, $Ai(s^{2/3}\eta)$ is decreasing, $Bi(s^{2/3}\eta)$ is increasing, and their product tends to zero. For $\eta < 0$, both functions are bounded. Thus there exists a constant M, independent of s, μ and the functions $g(\mu)$, such that

$$\left| Ai(s^{2/3}\eta) \right| \max_{0 \le y_1 \le y} \left| Bi(s^{2/3}\eta_1) \right| \le M ,$$

$$\left| Bi(s^{2/3}\eta) \right| \max_{y_1 \ge y} \left| Ai(s^{2/3}\eta_1) \right| \le M$$

for all $y \ge 0$. Hence (5.4) has the upper bound

$$\frac{\pi M}{s^{2/3}} \int_0^\infty |q^*\chi^*| dy . \tag{5.5}$$

We can now apply this estimate in two different ways. First, (5.3) and (5.5) give

$$|\chi^*(y)| \le |Ai(s^{2/3}\eta)| + \frac{\pi M}{s^{2/3}} \int_0^\infty |q^*\chi^*| dy ,$$

and if this is multiplied by $|q^*(y)|$ and integrated over $(0, \infty)$ it follows that

$$\int_0^\infty |q^*\chi^*| dy \le \left(1 - \frac{\pi M K}{s^{2/3}}\right)^{-1} \int_0^\infty |Ai(s^{2/3}\eta)\, q^*| dy ,$$

provided that $s^{2/3} > \pi MK$; consequently

$$\int_0^\infty |q^* \chi^*| \, dy \leq M'K \left(1 - \frac{\pi MK}{s^{2/3}}\right)^{-1} , \tag{5.6}$$

where M' is the upper bound of the absolute constant $|Ai(s^{2/3}\eta)|$.

Second, we have from (5.3) and (5.5)

$$|\chi^*(y) - Ai(s^{2/3}\eta)| \leq \frac{\pi M}{s^{2/3}} \int_0^\infty |q^* \chi^*| dy ,$$

and then it follows finally from (5.6) that, for all $y \geq 0$,

$$|\chi^*(y) - Ai(s^{2/3}\eta)| \leq \frac{\pi M M'K}{s^{2/3} - \pi MK} . \tag{5.7}$$

6. Asymptotic formulas for the eigenvalues

Let us now take the boundary condition at $y = 0$ to be $\chi = 0$. Then also $\chi^* = 0$, and so (5.7) becomes, for $y = 0$,

$$\left|Ai(s^{2/3}\eta_0)\right| \leq \frac{\pi M M'K}{s^{2/3} - \pi MK} , \tag{6.1}$$

where, by (4.2),

$$\frac{2}{3}(-\eta_0)^{3/2} = \int_0^{y^*} \left[\mu - g(y)\right]^{1/2} dy , \quad \mu = g(y^*) . \tag{6.2}$$

The inequality (6.1) involves only s and μ; on the right-hand side M and M' are fixed numbers and K is a function of μ which, since it was assumed that (5.3) converges uniformly in μ in any bounded μ-interval, is a continuous function of μ. It was noted at the beginning of Section 4 that, necessarily, $\mu \geq g(0)$. Let us therefore suppose that $g(0) \leq \mu \leq g(0) + A$, where A is a

positive constant. Then if in (6.1) K is replaced by its maximum in the numerator and by its minimum in the denominator, this inequality becomes

$$\left|\mathrm{Ai}(s^{2/3}\eta_o)\right| \leq \frac{\pi M M' K_1}{s^{2/3} - KMK_2} ,$$

where K_1, K_2 are fixed. If $s_o > (\pi MK_1)^{3/2})$ then there exists a constant B such that

$$\left|\mathrm{Ai}(s^{2/3}\eta_o)\right| \leq B\, s^{-2/3} \qquad (s \geq s_o) \tag{6.3}$$

As $s \to \infty$, the left-hand side of (6.3) is $O(s^{-1/6})$; hence we must take $s^{2/3}\eta_o$ to be close to a zero of the Airy function when s is large. These zeros may be denoted by $-a_1$, $-a_2$,..., where $a_1 < a_2 < \ldots$; thus

$$-\eta_o = a_n s^{-2/3} + O(s^{-2/3}) . \tag{6.4}$$

The constant implied by the symbol O depends on the order of n of the zero, but this can be ignored if we fix our attention on a particular zero.

Our final step is now to deduce an inequality for μ from (6.4) and (6.2). These two equations give

$$\int_o^{y^*} \left[\mu - g(y)\right]^{1/2} dy = \frac{2}{3} \frac{a_n^{3/2}}{s} + O(s^{-2/3}) .$$

Now g(y) is by hypothesis an increasing function of y; hence we may put $g(y) - g(0) = \zeta$, $y = h(\zeta)$ and write the above equation as

$$\int_o^{\mu - g(0)} \left[\mu - g(0) - \zeta\right]^{1/2} h'(\zeta)\, d\zeta = \frac{2}{3} \frac{a_n^{3/2}}{s} + O(s^{-2/3}) . \tag{6.5}$$

If we now suppose that

$$h'(0) = \frac{1}{g'(0)} \neq 0 , \tag{6.6}$$

then (6.5) is equivalent to

$$\frac{2}{3}\left[\mu - g(0)\right]^{3/2} \frac{1}{g'(0)} + 0\left\{\left[\mu - g(0)\right]^{5/2}\right\} = \frac{2}{3}\frac{\alpha_n^{3/2}}{s} + O(s^{-5/3}) ,$$

whence

$$\mu = \mu_n = g(0) + \left[g'(0)\right]^{2/3} \alpha_n s^{-2/3} + O(s^{-4/3})$$

or, by (4.1)

$$\lambda_n = g(0)\, s^2 + \left[g'(0)\right]^{2/3} \alpha_n s^{4/3} + O(s^{2/3}) . \tag{6.7}$$

This is the required approximation for λ_n, valid as $s \to \infty$ for fixed n.

Two remarks must be made about this result. The first is that by starting with an iterated form of the integral equation (5.3) a better approximation for λ_n can be obtained. It appears that λ_n can be expanded as an asymptotic series in decreasing powers of $s^{2/3}$. A formal method for obtaining its coefficients is presented in Appendix I.

Secondly, if $g(y) - g(0) \sim g_1 y^p$ $(p \neq 1)$, as $y \to 0$, then a formula for λ_n different from (6.7) is obtained. This point is also discussed in Appendix I.

If the boundary condition at $y = 0$ is $\chi' = 0$, a similar argument shows that the n-th eigenvalue is given by the same formula, (6.7), except that the α_n must be replaced by the zeros α_1', α_2', ... of $\mathrm{Ai}'(-\alpha)$.

7. Asymptotic formulas for the eigenfunctions

We can now derive asymptotic formulas for the functions of s which appear in a typical term of (3.13). For this purpose we write

$$\frac{\lambda}{s^2} = \mu = g(y^*) = g(0) + \left[g'(0)\right]^{2/3} \alpha\, s^{-2/3} + O(s^{2/3}) \tag{7.1}$$

withour specifying whether α is α_n or α_n'. Then by reversing the argument of the preceding section we obtain

$$\eta_o = -\alpha\, s^{-2/3} + O(s^{-4/3}) \ . \tag{7.2}$$

From now on, we shall omit the error terms, and write for example

$$\chi(y) \sim \left(\frac{d\eta}{dy}\right)^{-1/2} \mathrm{Ai}(s^{2/3}\eta) \ . \tag{7.3}$$

We begin by deriving expressions for $\chi(0)$, $\chi'(0)$ and ω_n. In the first place

$$\left[\frac{d\eta}{dy}\right]_{y=0} = \left(\frac{g(0) - \mu}{\eta_o}\right)^{1/2} \sim \left[\frac{\left[g'(0)\right]^{2/3} \alpha\, s^{-2/3}}{\alpha\, s^{-2/3}}\right]^{1/2}$$

so that

$$\left[\frac{d\eta}{dy}\right]_{y=0} \sim \left[g'(0)\right]^{1/3} \tag{7.4}$$

and hence

$$\chi(0) \sim \left[g'(0)\right]^{-1/3} \mathrm{Ai}(-\alpha) \ . \tag{7.5}$$

Also,

$$\chi'(y) \sim s^{2/3} \left(\frac{d\eta}{dy}\right)^{1/2} \mathrm{Ai}'(s^{2/3}\eta) - \frac{1}{2}\left(\frac{d\eta}{dy}\right)^{-3/2} \frac{d^2\eta}{dy^2} \mathrm{Ai}(s^{2/3}\eta) \ .$$

But, by (4.3), (7.1) and (7.2),

$$\frac{d^2\eta}{dy^2} = -\frac{g'(y)}{2\left[\mu - g(y)\right]^{1/2}(-\eta)^{1/2}} + \frac{\left[g(y) - \mu\right]^{1/2}}{2(-\eta)^{3/2}} \left[\frac{\mu - g(y)}{-\eta}\right]^{1/2}$$

$$= -\frac{g'(0)}{2\left[g'(0)\right]^{1/3} \alpha\, s^{-2/3}} \left[1 + O(s^{-2/3})\right] + \frac{1}{2} \frac{\left[g'(0)\right]^{2/3}}{\alpha\, s^{-2/3}} \left[1 + O(s^{-2/3})\right]$$

$$= O(1) \quad .$$

Hence

$$\chi'(0) \sim s^{2/3} \left[g'(0)\right]^{1/3} \mathrm{Ai}'(-\alpha) \quad . \tag{7.6}$$

Now, since λ is defined by (7.1), we have for example,

$$\frac{\partial}{\partial\alpha} \chi(0,s,\lambda) = \left[g'(0)\right]^{2/3} s^{4/3} \frac{\partial}{\partial\lambda} \chi(0,s,\lambda) + O(s^{-4/3}) \quad .$$

Hence ω_n, which is defined by (3.5), is given by

$$\omega_n \sim -\left[g'(0)\right]^{-1/3} \mathrm{Ai}(-\alpha_n) \left[g'(0)\right]^{-2/3} s^{4/3} \left[g'(0)\right]^{1/3} s^{2/3} \mathrm{Ai}''(-\alpha_n)$$

$$+ s^{2/3} \left[g'(0)\right]^{1/3} \mathrm{Ai}'(-\alpha_n) \left[g'(0)\right]^{-2/3} s^{-4/3} \left[g'(0)\right]^{-1/3} \mathrm{Ai}'(-\alpha_n)$$

and so, by (4.5),

$$\omega_n \sim s^{-2/3} \left[g'(0)\right]^{-2/3} \left\{\left[\mathrm{Ai}'(-\alpha_n)\right]^2 + \alpha_n \left[\mathrm{Ai}(-\alpha_n)\right]^2\right\} \quad . \tag{7.7}$$

In order to obtain $\chi(y,s,\lambda_n)$, we require an approximation for η when $y > y^*$. From (7.1), y^* is of order $s^{-2/3}$, and so for fixed y ($y \neq 0$) and sufficiently large s, χ is given by (7.3), where

$$\frac{2\eta^{3/2}}{3} = \int_{y^*}^{y} \left[g(y') - \mu\right]^{1/2} \, dy' \, .$$

If we write, in a notation slightly different from that of Section 6, $g(y) = \zeta$, $y = h(\zeta)$, this becomes

$$\frac{2\eta^{3/2}}{3} = \int_{0}^{g(y)} (\zeta - \mu)^{1/2} \, h'(\zeta) \, d\zeta$$

where μ is given by (7.1). Here $\frac{2\eta^{3/2}}{3}$, considered as a function of μ, is continuous for $\mu = g(0)$, and its derivatives can be obtained by differentiating with respect to μ under the integral sign, provided that $h'(\zeta)$ is sufficiently regular and that the resulting expressions for derivatives of order higher than the first are interpreted as 'finite parts' in the sense of Hadamard. Hence, by Taylor's theorem,

$$\frac{2\eta^{3/2}}{3} = \int_{0}^{g(y)} \left[\zeta - g(0)\right]^{1/2} h'(\zeta) \, d\zeta - \frac{1}{2}\left[g'(0)\right]^{2/3} \alpha \, s^{-2/3} \int_{0}^{g(y)} \left[\zeta - g(0)\right]^{-1/2} h'(\zeta) \, d\zeta$$

$$+ \, O(s^{-4/3}) \, ,$$

whence

$$\frac{2\eta^{3/2}}{3} = \int_{0}^{y} \left[g(y') - g(0)\right]^{1/2} dy' - \frac{1}{2}\left[g'(0)\right]^{2/3} \alpha s^{2/3} \int_{0}^{y} \frac{dy'}{\left[g(y') - g(0)\right]^{1/2}} + O(s^{-4/3}) \tag{7.8}$$

Now when s is large, $s^{2/3}\eta$ is large, and so (7.3) can be replaced by

$$\chi(y) \sim \left(\frac{d\eta}{dy}\right)^{-1/2} \frac{1}{2\pi^{1/2} \, s^{1/6} \, \eta^{1/4}} \exp\left[-\frac{2}{3} s \, \eta^{3/2}\right]$$

$$= \frac{1}{2\pi^{1/2} \left[g(y) - \mu\right]^{1/4} s^{1/6}} \exp\left[-\frac{2}{3} s \, \eta^{3/2}\right]$$

whence, by (7.8)

$$\chi(y) \sim \frac{1}{2\pi^{1/2}\left[g(y) - g(0)\right]^{1/4} s^{1/6}} \exp\left\{- s \int_0^y \left[g(y') - g(0)\right]^{1/2} dy'\right.$$

(7.9)

$$\left. + \frac{1}{2}\left[g'(0)\right]^{2/3} a\, s^{1/3} \int_0^y \frac{dy'}{\left[g(y') - g(0)\right]^{1/2}}\right\} .$$

Finally, we can obtain asymptotic formulas for $\Psi_n^{\pm}(x,s)$ and for $W_n(s)$ by applying the W.K.B. method to (3.9). We may take

$$\Psi_n^{\pm}(x,s) \sim \left[s^2 f(x) + \lambda_n\right]^{-1/4} \exp\left\{\mp \int_0^x \left[s^2 f(x') + \lambda_n\right]^{1/2} dx'\right\} , \tag{7.10}$$

whence

$$W_n(s) \sim 2 .$$

Also, by (6.7)

$$\int_0^x \left[s^2 f(x') + \lambda_n\right]^{1/2} dx' \sim s \int_0^x \left[f(x') + g(0)\right]^{1/2} dx'$$

$$+ \frac{1}{2} a_n \left[g'(0)\right]^{2/3} s^{1/3} \int_0^x \frac{dx'}{\left[f(x') + g(0)\right]^{1/2}} .$$

Hence

$$\frac{\Psi_n^{+}(x,s)\Psi_n^{-}(x_o,s)}{W_n(s)}$$

(7.11)

$$\sim \frac{\exp\left\{-s \int_0^x \left[f(x')+g(0)\right]^{1/2} dx' - \frac{1}{2} a_n g'(0)\, s^{1/3} \int_0^x \frac{dx'}{\left[f(x') + g(0)\right]^{1/2}}\right\}}{2s\left[f(x) + g(0)\right]^{1/4} \left[f(x_o) + g(0)\right]^{1/4}} .$$

8. Approximations for $\bar{\phi}$ and ϕ

It was shown in Section 3 that formally, for $x > x_o$,

$$(8.1) \qquad \bar{\phi} = \sum_{n=1}^{\infty} \bar{\phi}_n , \quad \bar{\phi}_n = \frac{\Psi_n^+(x,s)\, \Psi_n^-(x_o,s)\, \chi(y,s,\lambda_n)\, \chi(y_o,s,\lambda_n)}{W_n(s)\, \omega_n(s)} ,$$

and that a similar expression for $\bar{\phi}_n$ for $x < x_o$ is obtained by interchanging x and x_o. Suppose first that both $y > 0$ and $y_o > 0$. Then we can substitute (7.11), (7.9) and (7.7) into the equation for $\bar{\phi}_n$. If we put

$$(8.2) \qquad \begin{aligned} \tau = & \left| \int_{x_o}^{x} \left[f(x') + g(0) \right]^{1/2} dx' \right| + \int_{o}^{y} \left[g(y') - g(0) \right]^{1/2} dy' \\ & + \int_{o}^{y_o} \left[g(y') - g(0) \right]^{1/2} dy' , \end{aligned}$$

$$(8.3) \qquad \begin{aligned} \delta = & \left| \int_{x_o}^{x} \left[f(x') + g(0) \right]^{-1/2} dx' \right| - \int_{o}^{y} \left[g(y') - g(0) \right]^{-1/2} dy' \\ & - \int_{o}^{y_o} \left[g(y') - g(0) \right]^{-1/2} dy' , \end{aligned}$$

$$(8.4) \qquad F(x) = \left[f(x) + g(0) \right]^{1/4} , \quad G(y) = \left[g(y) - g(0) \right]^{1/4}$$

$$(8.5) \qquad \varepsilon_n = \left[Ai'(-\alpha_n) \right]^2 + \alpha_n \left[Ai(-\alpha_n) \right]^2 ,$$

then we have

$$(8.6) \qquad \bar{\phi}_n = \frac{\left[g'(0) \right]^{2/3}}{8\pi\, \varepsilon_n F(x) F(x_o) G(y) G(y_o)\, s^{2/3}} \exp \left\{ -s\tau - \frac{1}{2} \left[g'(0) \right]^{2/3} \alpha_n \delta\, s^{1/3} \right\} \left\{ 1 + O(s^{-1/3}) \right\} ,$$

where the error term is $O(s^{-1/3})$ because it is of the same order as the error terms in the expansions of the arguments of the exponentials in (7.9) and (7.11). It can be shown that (8.6) holds, not only for real positive s, but also for all s satisfying Re(s) > 0.

The approximation (8.6) cannot be substituted in (8.1) indiscriminately. In the first place, the series (8.1) will not be a solution of the Laplace-transformed problem unless it converges. It can be shown to be convergent (except when $x = x_o$) for real positive s. If it also converges on some line parallel to the imaginary axis in the region Re(s) > 0, and is dominated by a function which has an inverse Laplace transform defined by the usual complex inversion formula, then term-by-term inversion is legitimate. Now the example f = 0, g = 1 + y which is discussed in Appendix II shows that the series may not converge for Re(s) > 0 in the whole xy-plane. But this example also suggests that term-by-term evaluation may be permissible in a physically important region, which may be called the 'deep shadow'; this region includes a part of the boundary y = 0 which lies in the shadow. (The exact definition of the shadow will be given in the next section.)

We now proceed on the assumption that term-by-term inversion is permissible. In order to use (8.6) to derive an approximation for $\bar{\phi}$ and then one for ϕ, we must consider to what extent the asymptotic behavior of $\bar{\phi}$ can be inferred from that of the $\bar{\phi}_n$. Obviously, the factor $e^{-s\tau}$ in (8.6) is a 'delay factor', that is to say ϕ_n is obtained by replacing the variable t by $t - \tau$ in the inverse Laplace transform of $e^{s\tau}\,\bar{\phi}_n$. Since $\tau > 0$, then $e^{s\tau}\,\bar{\phi}_n$ must be a Laplace transform, and, a fortiori, it must tend to zero as $\mathrm{Re}(s) \to \infty$. Hence (8.6) can be used only when

(8.7) $$\delta > 0 .$$

It will be shown in the next section that this inequality defines the shadow, so that our approximation is suitable for the estimation of the diffraction effect.

Finally, we observe that the ratio of the leading term of the n-th mode, with $n > 1$, to the error term of the first mode is

$$\frac{\varepsilon_1}{\varepsilon_n} \exp\left\{-\frac{1}{2}\left[g'(0)\right]^{2/3} (a_n - a_1)\,\delta\, s^{1/3}\right\} O(s^{-1/3}) \quad ,$$

which, because of (8.7), tends to zero as $|s| \to \infty$ in $\mathrm{Re}(s) > 0$. In other words, the error term of the first mode is ultimately greater than the leading term of any succeeding mode. Therefore we can retain the leading term of the first mode only when we use (8.6). We have then, for $y > 0$ $y_o > 0$, as $\mathrm{Re}(s) \to \infty$

$$(8.8) \qquad \bar{\phi} = \frac{\left[g'(0)\right]^{2/3}}{8\pi\, \varepsilon_1 F(x)F(x_o)G(y)G(y_o)} \frac{1 + O(s^{-1/3})}{s^{2/3}} \exp\left\{-s\tau - \frac{1}{2}\left[g'(0)\right]^{2/3} a_1 \delta\, s^{1/3}\right\},$$

If either $y \to 0$ or $y_o \to 0$, then the factors $1/G(y)$ or $1/G(y_o)$ respectively become infinite, and (8.8) must be modified. The requisite modification is different for the two boundary conditions at $y = 0$, $\phi = 0$ or $\partial\phi/\partial y = 0$. Suppose first that $\partial\phi/\partial y = 0$ at $y = 0$. Then $\phi \neq 0$ if either or both y and y_o are zero. Suppose next that $y = 0$. Then the term $\chi(0,s,\lambda_n)$ in $\bar{\phi}_1$ must be replaced not by (7.9) but by (7.5), and (8.8) must be replaced by

$$(8.9) \qquad \bar{\phi} = \frac{\left[g'(0)\right]^{1/3}}{4\pi^{1/2} a_1' \mathrm{Ai}(-a_1') F(x)F(x_o)G(y_o)} \frac{1 + O(s^{-1/3})}{s^{1/2}} \exp\left\{-s\tau - \frac{1}{2}\left[g'(0)\right]^{2/3} a_1' \delta s^{1/3}\right\},$$

$$(y = 0,\ y_o > 0)$$

where α_1' is the first zero of $\mathrm{Ai}'(-\gamma)$. [The quantities τ and δ retain the meanings defined in (8.2), (8.3).] It may be noted that (8.9) can be obtained from (8.8) by replacing the factor $1/G(y)$ by $2\pi^{1/2}\,\mathrm{Ai}(-\alpha_1')\,[g'(0)]^{-1/3}\,s^{1/6}$.

If $y > 0$, $y_o = 0$, then $\bar{\phi}$ is given by (8.9) with y_o replaced by y. If $y_o = 0$ in addition to $y = 0$, then $\bar{\phi}$ can be obtained from (8.9) by replacing $1/G(y_o)$ by $2\pi^{-1/2}\,\mathrm{Ai}(-\alpha_1')\,[g'(0)]^{-1/3}\,s^{-1/6}$:

$$(8.10)\qquad \bar{\phi} = \frac{1}{2\alpha_1'\,F(x)F(x_o)}\,\frac{1+O(s^{-1/3})}{s^{1/3}}\,\exp\left\{-s\tau - \tfrac{1}{2}\left[g'(0)\right]^{2/3}\alpha_1'\delta\,s^{1/3}\right\},$$

$$(y = y_o = 0)\ .$$

When the boundary condition at $y = 0$ is that $\bar{\phi} = 0$, then $\partial\bar{\phi}/\partial y$ can be evaluated approximately from the leading term of $\partial\bar{\phi}_1/\partial y$ by means of (7.11), (7.9), (7.7) and (7.6): we have

$$(8.11)\qquad \frac{\partial\bar{\phi}}{\partial y} = \frac{g'(0)}{4\pi^{1/2}\mathrm{Ai}'(-\alpha_1)F(x)F(x_o)G(y_o)}\left[s^{1/6}+O(1)\right]\exp\left\{-s\tau - \tfrac{1}{2}\left[g'(0)\right]^{2/3}\alpha_1\delta s^{1/3}\right\}$$

$$(y = 0,\ y_o > 0)\ ,$$

where α_1 is the first zero of $\mathrm{Ai}(-\gamma)$. When $y_o = 0$ with this same boundary condition, we have the trivial result $\bar{\phi} = 0$, since each $\bar{\phi}_n$ then contains a zero factor. But $\partial\bar{\phi}/\partial y_o$ retains a meaning; it is the field due to an 'instantaneous doublet' at $x = x_o$, $y = 0$. The form of an approximation for this field can be deduced by replacing y_o by y in (8.11). For $y = 0$ one can then evaluate $\partial^2\bar{\phi}/\partial y\partial y_o$; it is given by

$$(8.12)\qquad \frac{\partial^2\bar{\phi}}{\partial y\partial y_o} = \frac{\left[g'(0)\right]^{4/3}}{2F(x)F(x_o)}\left[s+O(s^{2/3})\right]\exp\left\{-s\tau - \tfrac{1}{2}\left[g'(0)\right]^{2/3}\alpha_1\delta\,s^{1/3}\right\},$$

$$(y = y_o = 0)\ .$$

9. The geometrical optics of the problem

The inverse Laplace transforms of the approximations (8.8) - (8.12) can now be written down explicitly. Before doing this, however, we propose to consider the geometrical optics of the problem, as this will enable us to give simple interpretations of the quantities τ, δ and the factors $\left[F(x)\right]^{-1}$, ..., $\left[\bar{G}(y_o)\right]^{-1}$.

It can be shown, by a simple modification of the proof given previously[2] that the arrival time of the disturbance at any point can be deduced from Fermat's principle suitably modified when there is diffraction. For the moment let $t = \tau(x,y)$ be the equation of the front of any disturbance governed by our wave equation (1.1). Then τ satisfies the eikonal equation

$$\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2 = f(x) + g(y) \quad . \tag{9.1}$$

The rays associated with the fronts $t = \tau$ are the characteristics of (9.1), and are defined by the ordinary differential equations

$$\frac{dx}{d\sigma} = \frac{d\tau}{dx}, \quad \frac{dy}{d\sigma} = \frac{\partial\tau}{\partial y} \quad , \tag{9.2}$$

where σ is a parameter. From (9.2) we can deduce that

$$\frac{d\tau}{d\sigma} = \frac{\partial\tau}{\partial x}\frac{dx}{d\sigma} + \frac{\partial\tau}{\partial y}\frac{dy}{d\sigma} = \left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2 \quad ,$$

or, by (9.1)

$$\frac{d\tau}{d\sigma} = f(x) + g(y) \quad . \tag{9.3}$$

Again, we have for example

$$\frac{d}{d\sigma}\left(\frac{\partial\tau}{\partial x}\right) = \frac{\partial^2\tau}{\partial x^2}\frac{dx}{d\sigma} + \frac{\partial^2\tau}{\partial x\partial y}\frac{dy}{d\sigma} = \frac{\partial^2\tau}{\partial x^2}\frac{\partial\tau}{\partial x} + \frac{\partial^2\tau}{\partial x\partial y}\frac{\partial\tau}{\partial y}$$

$$= \frac{1}{2}\frac{\partial}{\partial x}\left\{\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2\right\},$$

whence, by (9.1)

$$(9.4)\qquad \frac{d}{d\sigma}\left(\frac{\partial\tau}{\partial x}\right) = \frac{1}{2}f'(x) \qquad \frac{d}{d\sigma}\left(\frac{\partial\tau}{\partial y}\right) = \frac{1}{2}g'(y)\ .$$

The rays are also the geodesics associated with the line element

$$(9.5)\qquad \left\{f(x) + g(y)\right\}\left\{dx^2 + dy^2\right\} = d\tau^2\ ,$$

where the identification with $d\tau^2$ is an immediate consequence of (9.1), (9.2) and (9.3), so that τ is the geodesic distance measured along a ray.

The integration of (9.2) and (9.4) can be effected by quadratures. First, we have

$$\frac{d\tau}{dx}\frac{d}{d\sigma}\left(\frac{\partial\tau}{\partial x}\right) = \frac{1}{2}f'(x)\frac{dx}{d\sigma}, \qquad \frac{\partial\tau}{\partial y}\frac{d\tau}{d\sigma}\left(\frac{\partial\tau}{\partial y}\right) = \frac{1}{2}g'(y)\frac{dy}{d\sigma}\ ,$$

so that

$$\left(\frac{\partial\tau}{\partial x}\right)^2 - f(x) = \text{const.}, \qquad \left(\frac{\partial\tau}{\partial y}\right)^2 - g(y) = \text{const.}$$

Here $\partial\tau/dx$, $\partial\tau/\partial y$ can be replaced by $dx/d\sigma$, $dy/d\sigma$ respectively, and since this can also be done in (9.1) it follows at once that we can write

$$(9.6)\qquad \left(\frac{dx}{d\sigma}\right)^2 = f(x) + \mu \qquad \left(\frac{dy}{d\sigma}\right)^2 = g(y) - \mu\ ,$$

where μ is an arbitrary constant. Also, we can write (9.3) as

$$\tau = \int\left[f(x) + \mu\right]d\sigma + \int\left[g(y) - \mu\right]d\sigma$$

or

$$\tau = \pm \int \left[f(x) + \mu\right]^{1/2} dx + \int \left[g(y) - \mu\right]^{1/2} dy \quad . \tag{9.7}$$

This holds along any arc of a ray on which $dx/d\sigma$, $dy/d\sigma$ are of constant sign, and the sign in (9.7) must be chosen such that both terms are positive.

Now consider the rays from (x_o, y_o), the position of the instantaneous source. The equations of these rays are obtained by integrating (9.6) for say, $x > x_o$, $y < y_o$:

$$\sigma = \int_{x_o}^{x} \frac{dx'}{\left[f(x') + \mu\right]^{1/2}} = \int_{y}^{y_o} \frac{dy'}{\left[g(y') - \mu\right]^{1/2}} \quad . \tag{9.8}$$

(For $x < x_o$, the sign of the first integral must be reversed; and for $y > y_o$, the sign of the second integral must be reversed.)

Now if $dy/d\sigma$ is negative at $x = x_o$, $y = y_o$, then y decreases at first, and there are three possibilities. If $\mu < g(0)$, the ray meets $y = 0$ at a finite angle. It is then continued as a reflected ray. Defining the 'reflection region' as consisting of the points in the plane which lie on a reflected ray, we can say that at a point in this region the disturbance curves along a direct ray and remains identical with the Hadamard elementary solution of (1.1) until the time t is equal to the geodesic distance τ measured along the reflected ray and the corresponding incident ray. After that, a reflected pulse must be added. This pulse cannot be calculated from the series of modes; it can be obtained either by converting the series for $\bar{\phi}$ into a certain integral and evaluating this as an asymptotic series by the method of steepest descent, or directly by expanding the reflected pulse as a series in half-integral powers of $t^2 - \tau^2$, where τ is the arrival time of the reflected pulse.

If, on the other hand, $\mu > g(0)$, then the ray becomes parallel to $y = 0$ at a distance y^* from $y = 0$ given by $g(y^*) = \mu$. Subsequently, the ray turns away from $y = 0$ again.

The third possibility, the transition case $\mu = g(0)$, gives the glancing ray. If the shadow is defined as that part of the plane which cannot be reached along a direct or reflected ray from (x_o, y_o) then the shadow boundary may consist of the glancing ray beyond its point of contact with $y = 0$, or it may consist of a portion of the glancing ray and of the envelope of the incident rays. But in any case no reflected ray can cross the glancing ray.

Now, let $(x_1, 0)$ be the point at which the glancing ray touches $y = 0$. Then by (9.8) $\sigma = \sigma_1$ where, since $\mu = g(0)$,

$$(9.9) \qquad \sigma_1 = \int_{x_o}^{x_1} \frac{dx}{\left[f(x) + g(0)\right]^{1/2}} = \int_{0}^{y_o} \frac{dy}{\left[g(y) - g(0)\right]^{1/2}} .$$

Let $x_2 > x_1$, and consider a point (x,y) in the shadow region on the ray touching $y = 0$ at $x = x_2$, with $x > x_2$ (see Figure 1). Let σ_2 be the

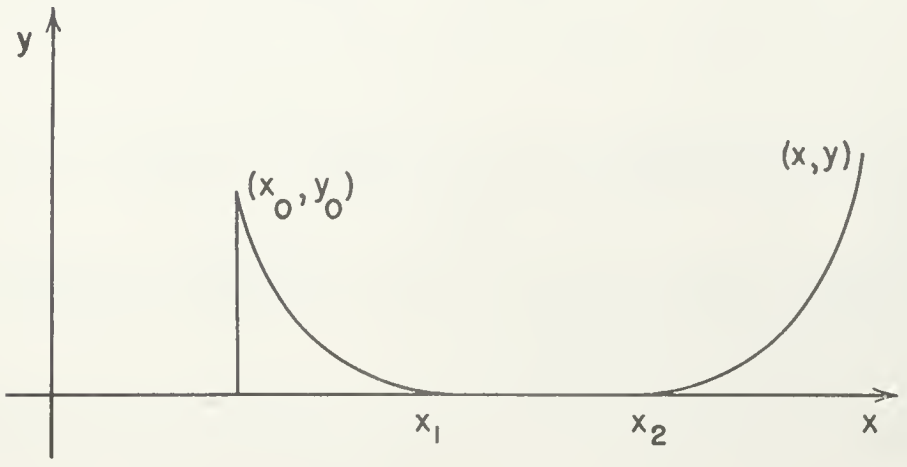

Figure 1

difference between the values of σ at (x,y) and at $(x_2,0)$. Then clearly

$$(9.10) \qquad \sigma_2 = \int_{x_2}^{x} \frac{dx'}{\left[f(x') + g(0)\right]^{1/2}} = \int_{0}^{y} \frac{dy'}{\left[g(y') - g(0)\right]^{1/2}} .$$

Now by (8.3) we have, for $x > x_o$,

$$\delta = \int_{x_o}^{x} \frac{dx'}{\left[f(x') + g(0)\right]^{1/2}} - \int_{0}^{y} \frac{dy'}{\left[g(y') - g(0)\right]^{1/2}} - \int_{0}^{y_o} \frac{dy'}{\left[g(y') - g(0)\right]^{1/2}} .$$

Hence, by (9.9) and (9.10),

$$(9.11) \qquad \delta = \int_{x_1}^{x_2} \frac{dx'}{\left[f(x') + g(0)\right]^{1/2}} .$$

This equation furnishes the interpretation of δ, and shows that $\delta > 0$ implies that (x,y) cannot be reached by a reflected ray.

Next, we can calculate the arrival time of the diffracted wave at (x,y); it is equal to the geodesic distance between (x,y) and (x_o,y_o) measured along the rays from (x_o,y_o) to $(x_1,0)$ and from $(x_2,0)$ to (x,y) and along the boundary $x_1 \le x \le x_2$, $y = 0$. Hence by (9.7) and (9.5), the arrival time is

$$\int_{x_o}^{x_1} \left[f(x') + g(0)\right]^{1/2} dx' + \int_{0}^{y_o} \left[g(y') - g(0)\right]^{1/2} dy + \int_{x_1}^{x_2} \left[f(x') + g(0)\right]^{1/2} dx'$$

$$+ \int_{x_2}^{x} \left[f(x') + g(0)\right]^{1/2} dx' + \int_{0}^{y} \left[g(y') - g(0)\right]^{1/2} dy' = \tau .$$

This is the quantity τ defined by (8.2). If $x < x_o$, similar considerations apply, and lead to analogous conclusions.

Finally, we may observe that the factor $\left[F(x)F(x_o)G(y)G(y_o)\right]^{-1}$ can

also be related to the geometry of the rays. We take $x > x_0$ and use the notations in (8.2) and (8.4), and let

$$D(x,y) = \left[F(x)G(y)\right]^{-1} . \tag{9.12}$$

Then it is easily verified that

$$2\left(\frac{\partial \tau}{\partial x}\frac{\partial D}{\partial x} + \frac{\partial \tau}{\partial y}\frac{\partial D}{\partial y}\right) + D\nabla^2\tau = 0 . \tag{9.13}$$

This is the transport equation associated with the front $t = \tau$ (see [4], p 24). From it one can deduce, by an argument similar to the one used for the case of a homogeneous medium, (see [1]), that the product D^2 times the refractive index $\left[f(x) + g(y)\right]^{1/2}$, times the normal cross-section is constant along a 'tube' formed by adjacent diffracted rays. The fact that $D \to \infty$ as $y \to 0$ is an immediate consequence of the fact that $y = 0$ is a caustic of the diffracted rays.

10. Approximations for ϕ

The inverse Laplace transforms of (8.8) - (8.12) can be evaluated in terms of the functions U_k defined in the author's paper on diffraction by a cylinder [3]. These are defined by

$$U_k(\gamma,T) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s^{-k}\exp\left(s\,T - \gamma\, s^{1/3}\right)\,ds . \tag{10.1}$$

U_k is zero for $T < 0$, and is given for small positive T by the approximation

$$U_k(\gamma,T) \sim \frac{3^{(6k-1)/4}}{2\pi^{1/2}}\,\gamma^{(3-6k)/4}\,T^{(6k-5)/4}\,\exp\left(-\frac{2\gamma^{3/2}}{3^{3/2}\,T^{1/2}}\right) . \tag{10.2}$$

Also,

$$(10.3)\quad U_{2/3}(\gamma,T) = 3^{2/3}\,T^{-1/3}\,H(T)\,\mathrm{Ai}\left\{\gamma(3T)^{-1/3}\right\} = \frac{1}{\pi}\left(\frac{1}{T}\right)^{1/2} K_{1/3}\left(\frac{2\,\gamma^{3/2}}{3^{3/2}\,T^{1/2}}\right) H(T)$$

and

$$(10.4)\qquad \frac{\partial U_k(\gamma,T)}{\partial\gamma} = -\,U_{(3k-1)/3}(\gamma,T)\ .$$

The formula (8.8) can be written, in terms of (9.12), as

$$\bar{\phi} \sim \frac{\left[g'(0)\right]^{2/3}}{8\pi\,\varepsilon_1}\,D(x,y)D(x_o,y_o)s^{-2/3}\exp\left\{-s\tau\pi - \frac{1}{2}\left[g'(0)\right]^{2/3}\alpha_1\,\delta s^{1/3}\right\}\ .$$

Hence, from (10.3),

$$(10.5)\quad \phi \sim \frac{g'(0)}{8\pi^2\varepsilon_1}\,D(x,y)D(x_o,y_o)\left(\frac{\alpha_1\delta}{T}\right)^{1/2} K_{1/3}\left\{\frac{g'(0)(\alpha_1\delta)^{3/2}}{6(3T)^{1/2}}\right\} H(T)\ ,$$

where

$$(10.6)\qquad T = t - \tau$$

is the time counted from the arrival of the diffracted front. This is valid for small T, and can be replaced for sufficiently small T, by the crude approximation

$$(10.7)\quad \phi \sim \frac{3^{3/4}\left[g'(0)\right]^{1/2}}{16\pi^{3/2}\,\varepsilon_1}\,D(x,y)D(x_o,y_o)\left(\frac{2}{\alpha_1\delta T}\right)^{1/4}\exp\left\{-\frac{g'(0)(\alpha_1\delta)^{3/2}}{6(3T)^{1/2}}\right\}\ .$$

In this formula, ε_1 and α_1 are absolute constants; δ,τ (which enters through T) and $D(x,y)$ are functions of position which can be derived from geometrical optics; thus only $g'(0)$ appears as a parameter specifically connected with the particular form, $f(x) + g(y)$, that we have assumed for the square of the refractive index of the medium.

The corresponding approximations derived from (8.9) - (8.12) are as follows. If the boundary condition at $y = 0$ is that $\partial\phi/\partial y = 0$, then from (8.9), (8.10),

$$(10.8)\quad \phi \sim \frac{3^{1/2}\left[g'(0)\right]^{1/3}}{8\pi^{1/2}\,a_1'\mathrm{Ai}(-a_1')}\,\frac{D(x_o,y_o)}{F(x)}\,T^{-1/2}\exp\left\{-\frac{g'(0)(a_1'\delta)^{3/2}}{6(3T)^{1/2}}\right\},\qquad y = 0,\ y_o>0;$$

$$(10.9)\quad \phi \sim \frac{3^{1/4}\left[g'(0)\right]^{1/6}}{4\pi^{1/2}\,a_1'\,F(x)F(x_o)}\left(\frac{a_1'\delta}{2T^3}\right)^{1/4}\exp\left\{-\frac{g'(0)(a_1'\delta)^{3/2}}{6(3T)^{1/2}}\right\},\qquad y = y_o = 0\,.$$

If on the other hand $\phi = 0$ at $y = 0$, then (8.11), (8.12) give

$$(10.10)\quad \frac{\partial\phi}{\partial y} \sim \frac{\left[g'(0)\right]^{4/3}}{8\pi\,\mathrm{Ai}'(-a_1)}\,\frac{D(x_o,y_o)}{F(x)}\left(\frac{a_1\delta}{2}\right)^{1/2}\frac{1}{T}\exp\left\{-\frac{g'(0)(a_1\delta)^{3/2}}{6(3T)^{1/2}}\right\},\qquad y = 0,\ y_o>0,$$

$$(10.11)\quad \frac{\partial^2\phi}{\partial y_o\partial y} \sim \frac{3\left[g'(0)\right]^{-1/6}}{4\pi^{1/2}\,F(x)F(x_o)}\left(\frac{2}{a_1\delta}\right)^{3/4}(3T)^{1/4}\exp\left\{-\frac{g'(0)(a_1\delta)^{3/2}}{6(3T)^{1/2}}\right\}\,.$$

Appendix I. Asymptotic Expansion of the Eigenvalues

The problem considered in Section 6 was that of determining the behavior, as $s \to \infty$, of the eigenvalues λ_n of the differential equation

$$\frac{d^2\chi}{dy^2} + \left[\lambda - s^2 g(y)\right]\chi = 0 \tag{I.1}$$

subject to the conditions, $\chi \to 0$ as $y \to \infty$ and $\chi = 0$ or $d\chi/dy = 0$ at $y = 0$. The object of this Appendix is to give an alternative method of attacking this problem; it is purely formal but as such very effective.

Consider first the simple example

$$g(y) = 1 + y^p \qquad (p \text{ real and positive}). \tag{I.2}$$

[it was assumed at the outset that $g(0) > 0$]. Then

$$\frac{d^2\chi}{dy^2} + \left[\lambda - s^2 - s^2 y^p\right]\chi = 0 .$$

Now let

$$\eta = y\, s^{2/(p+2)}, \quad \alpha = (\lambda - s^2)\, s^{4/(p+2)} ; \tag{I.3}$$

then we have, instead,

$$\frac{d^2\chi}{d\eta^2} + (\alpha - \eta^p)\chi = 0 ,$$

and this has a solution satisfying the boundary conditions at $\eta = 0$ and at $\eta = \infty$ only if $\alpha = \alpha_n$ when $\alpha_1, \alpha_2, \ldots$ are a certain set of numbers (which depends on p). Hence we can state at once that

$$\lambda_n = s^2 + \alpha_n\, s^{4/(p+2)} , \tag{I.4}$$

that is to say, we can determine λ_n explicitly as a function of s. If $p = 1$, then (I.4) reduces to (6.2) without the error term.

We can now try to deal with the general case by a similar change of the independent variable. For simplicity we suppose that

$$g(0) > 0, \quad g'(0) > 0, \quad g(y) = g_0 + g_1 y + g_n y^2 + \dots , \tag{I.5}$$

where $g_m = g^{(m)}(0)/m!$ and substitute (I.5) formally in (I.1). We also assume an expansion for λ:

$$\lambda = s^2 \sum_{m=0}^{\infty} c_m s^{-2m/3} , \qquad c_0 = g_0 \tag{I.6}$$

Then (I.1) becomes, in terms of η,

$$\eta = y s^{2/3} , \tag{I.7}$$

$$\frac{d^2\chi}{d\eta^2} + \left\{ \sum_{m=0}^{\infty} (c_{m+1} - g_{m+1} \eta^{m+1}) s^{-2m/3} \right\} \chi = 0 . \tag{I.8}$$

Now put

$$\chi = \sum_{m=0}^{\infty} Y_m(\eta) s^{-2m/3} \tag{I.9}$$

and substitute this in (I.8). Then one obtains, by equating the coefficients of equal powers of $s^{2/3}$ to zero, a system of recurrence equations:

$$\frac{d^2 Y_0}{d\eta^2} + (c_1 - g_1 \eta) Y_0 = 0 \tag{I.10}$$

$$\begin{cases} \dfrac{d^2 Y_1}{d\eta^2} + (c_1 - g_1 \eta) Y_1 = +(g_2 \eta^2 - c_2) Y_0 \\ \dots \qquad\qquad \dots \qquad\qquad \dots \\ \dfrac{d^2 y_m}{d\eta^2} + (c_n - g_n \eta) Y_m = \displaystyle\sum_{j=1}^{m} (g_{j+1} \eta^{j+1} - c_{j+1}) Y_{m-j} . \end{cases} \tag{I.11}$$

We can now impose on each Y_m the boundary conditions

(I.12) $\quad Y_m \to 0$ as $\eta \to \infty$; $\quad Y_m = 0$ (or $dY_m/d\eta = 0$) at $\eta = 0$.

(The 'turning point' at which $g(y) = \lambda/s^2$ presents no difficulty, as the substitutions (I.6) and (I.7) assure that its position is essentially independent of s for large s).

A solution of (I.10) which tends to zero as $\eta \to \infty$ is

$$Y_o = \mathrm{Ai}(g_1^{1/3}\eta - c_1 g_1^{-2/3}) \ .$$

Hence the conditions at $\eta = 0$ give

(I.13) $$c_1 = \alpha\, g_1^{2/3} \ ,$$

where $\mathrm{Ai}(-\alpha) = 0$ or $\mathrm{Ai}'(-\alpha) = 0$ respectively, according to which of the boundary condition at $\eta = 0$ obtains. Hence

(I.14) $$Y_o = L_o \mathrm{Ai}(g_1^{1/3}\eta - \alpha) \ ,$$

where L_o is a constant. The equation for Y_1 is now

(I.15) $$\frac{d^2Y_1}{d\eta^2} + (\alpha\, g_1^{2/3} - g_1\eta)Y_1 = L_o(g_2\eta^2 - c_2)\,\mathrm{Ai}(g_1^{1/3}\eta - \alpha) \ ,$$

whence

$$Y_1 = L_1 \mathrm{Ai}(g_1^{1/3}\eta - \alpha) + M_1 \mathrm{Bi}(g_1^{1/3}\eta - \alpha)$$

(I.16) $$+ \frac{\pi L_o}{g_1^{1/3}} \int_o^{\eta} \left[\mathrm{Ai}(g_1^{1/3}\eta' - \alpha)\mathrm{Bi}(g_1^{1/3}\eta - \alpha) - \mathrm{Ai}(g_1^{1/3}\eta - \alpha)\mathrm{Bi}(g_1^{1/3}\eta' - \alpha)\right]$$

$$\times \left[g_2\eta'^2 - c_2\right] \mathrm{Ai}(g_1^{1/3}\eta' - \alpha)\, d\eta' \ ,$$

where L_1, M_1 are constants. If for example $\chi = 0$ at $y = 0$, then for $\chi = 0$,

$$Y_1 = M_1 Bi(-a) = 0 ,$$

whence $M_1 = 0$; a similar argument holds if $d\chi/dy = 0$ at $y = 0$. Thus $M_1 = 0$. One can also take $L_1 = 0$, since in the subsequent calculation $L_1 \neq 0$ simply gives rise to a series similar to that which has the factor L_o, except that this one is multiplied by $L_1/s^{2/3}$. Then we can take $L_o = 1$.

Now as $\eta \to \infty$, the only non-vanishing term on the right-hand side of (I.16) is

$$\frac{\pi}{g_1^{1/3}} Bi(s_1^{1/3}\eta - a) \int_o^{\eta} (g_2 \eta'^2 - c_2)\left[Ai(g_1^{1/3}\eta' - a)\right]^2 d\eta' .$$

This can be written as

$$\frac{\pi L_o}{g_1^{1/3}} Bi(g_1^{1/3} - a)\left(\int_o^{\infty} - \int_{\eta}^{\infty}\right)(g_2 \eta'^2 - c_2)\left[Ai(g_1^{1/3}\eta' - a)\right]^2 d\eta' .$$

Hence a necessary and sufficient condition for Y_1 to approach infinity as $\eta \to \infty$ is that

$$\int_o^{\infty} (g_2 \eta'^2 - c_2)\left[Ai(g_1^{1/3}\eta' - a)\right]^2 d\eta' = 0 ,$$

and this determines c_2:

$$c_2 = \frac{\int_o^{\infty} g_2 \eta'^2 \left[Ai(g_1^{1/3}\eta' - a)\right]^2 d\eta'}{\int_o^{\infty} \left[Ai(g_1^{1/3}\eta' - a)\right]^2 d\eta'} . \qquad \text{(I.17)}$$

This argument can be repeated at each stage. In fact, we can take

$$Y_m = \frac{\pi}{g_1^{1/3}} \int_0^{\eta} \left[Ai(g_1^{1/3}\ \eta' - \alpha) Bi(g_1^{1/3}\ \eta - \alpha) - Ai(g_1^{1/3}\ \eta - \alpha) Bi(g_1^{1/3}\ \eta' - \alpha) \right]$$

(I.18)

$$\times \left[\left(g_{m+1}\ \eta'^{m+1} - c_{m+1} \right)\ Ai(g_1^{1/3}\ \eta' - \alpha) + \sum_{j=1}^{m-1} \left(g_{j+1}\ \eta'^{j+1} - c_{j+1} \right) Y_{m-j}(\eta') d\eta' \right]$$

and we have then

$$(I.19)\quad c_{m+1} = \frac{\int_0^{\infty} \left[g_{m+1}\ \eta'^{m+1} Ai(g_1^{1/3}\eta - \alpha) + \sum_{j=1}^{m-1} (g_{j+1}\ \eta^{j+1} - c_{j+1}) Y_{m-j} \right] Ai(g_1^{1/3}\eta - \alpha) d\eta}{\int_0^{\infty} \left[Ai(g_1^{1/3}\ \eta - \alpha) \right]^2 d\eta},$$

where $m = 2,3,\ldots$.

It is not clear, at this stage, when this method is valid; nor have we determined whether (I.6) is convergent for sufficiently large s, or whether it is an asymptotic expansion valid as $s \to \infty$.

Appendix II. Convergence of the series for $\bar{\phi}$ in a special case

The simplest example of a problem of the type considered in this paper is

$$f(x) = 0, \quad g(y) = 1 + y . \tag{II.1}$$

These functions satisfy the requirements (1.5) - (1.7). We can then take, for the functions χ defined in Section 3,

$$\chi(y,s,\lambda) = \mathrm{Ai}\left\{ s^{2/3} y - s^{-4/3} (\lambda - s^2) \right\} .$$

Then

$$\lambda_n = s^2 + a_n s^{4/3} , \tag{II.2}$$

where again either $\mathrm{Ai}(-a_n) = 0$ or $\mathrm{Ai}'(-a_n) = 0$; this is (6.7) without the error term. Thus

$$\chi(y,s,\lambda_n) = \mathrm{Ai}(s^{2/3} y - a_n)$$

and, by (3.5),

$$\omega_n = \varepsilon_n s^{-2/3}$$

where ε_n is defined by (8.5). Also, we can take, again in the notation of Section 3,

$$\Psi_n^+(x,s) = \exp\left[-x\sqrt{\lambda_n}\right], \quad \Psi_n^-(x,s) = \exp\left[x\sqrt{\lambda_n}\right], \quad W_n = 2\sqrt{\lambda_n} ,$$

and so (3.13) becomes

$$\bar{\phi} = \sum_{n=1}^{\infty} \frac{\mathrm{Ai}(-a_n + s^{2/3} y)\,\mathrm{Ai}(-a_n + s^{2/3} y_o)}{2\varepsilon_n (s^{2/3} + a_n)^{1/2}} \exp\left\{ - |x - x_o| (s^2 + a_n s^{3/4})^{1/2} \right\} . \tag{II.3}$$

In order to discuss this series, we must use the asymptotic behavior of the Airy function. This is given by the formulas

$$\text{(II.4)} \qquad \mathrm{Ai}(\gamma) = \frac{1}{2\pi^{1/2}\,\gamma^{1/4}} \exp\left(-\frac{2}{3}\gamma^{3/2}\right)\left\{1 + O(\gamma^{-3/2})\right\} \qquad (-\pi < \arg\gamma < \pi),$$

$$\text{(II.5)} \qquad \mathrm{Ai}(\gamma) = \frac{1}{2\pi^{1/2}\,\gamma^{1/4}} \exp\left(\frac{2}{3}\gamma^{3/2}\right)\left\{1 + O(\gamma^{-3/2})\right\} \qquad (\pi < \arg\gamma < 3\pi),$$

where $\gamma^{1/4}$, $\gamma^{3/2}$ are to be interpreted as principal values. For real positive α, $\mathrm{Ai}(-\alpha)$ is given by

$$\text{(II.6)} \qquad \mathrm{Ai}(-\alpha) = \frac{1}{\pi^{1/2}\,\alpha^{1/4}} \left\{\sin\left(\frac{2}{3}\alpha^{3/2} + \frac{\pi}{4}\right) + O(\alpha^{-3/2})\right\}.$$

From this one can deduce at once that the zeros a_n of $\mathrm{Ai}(-\alpha)$ are given by

$$\text{(II.7)} \qquad \frac{2}{3}a_n^{3/2} = (n - \tfrac{3}{4})\pi + O(\tfrac{1}{n}), \quad a_n = \left\{\frac{3\pi}{2}(n - \tfrac{3}{4})\right\}^{2/3} + O(n^{-4/3}) \qquad (n = 1,2,\ldots),$$

while those of $\mathrm{Ai}'(-\alpha)$ are given by

$$\text{(II.8)} \qquad \frac{2}{3}\left(a_n'\right)^{3/2} = (n - \tfrac{3}{4})\pi + O(\tfrac{1}{n}), \quad a_n' = \left\{\frac{3\pi}{2}(n - \tfrac{3}{4})\right\}^{2/3} + O(n^{-4/3}) \qquad (n = 1,2,\ldots).$$

Both these sets of formulas are valid as $n \to \infty$; the error terms are numerically small for $n \geq 2$.

We can now examine the convergence of the series (II.3). For real positive s, the series converges for all (x,y) except on the line $x = x_o$, since the Airy functions are then bounded. Next, suppose that $0 < |\arg s| \leq \frac{1}{2}\pi$. Then, for sufficiently large n (and given x,y,s) we have

$$\left|\arg\left(-a_n + s^{2/3}y\right)\right| < \pi, \quad \left|\arg\left(-a_n + s^{2/3}y_o\right)\right| < \pi,$$

so that, by (II.4), we have

$$\mathrm{Ai}(s^{2/3}y - a_n) \sim \frac{1}{2\pi^{1/2}(-a_n)^{1/4}} \exp\left\{-\frac{2}{3}(s^{2/3}y - a_n)^{3/2}\right\};$$

a similar expression holds for $\mathrm{Ai}(s^{2/3}y_o - \alpha_n)$. Now $s^{2/3}y - \alpha_n$ is, for large n, in the second or third quadrant, according as $\arg s > 0$, $\arg s < 0$ respectively. Hence

$$\frac{2}{3}(-\alpha_n + s^{2/3}y)^{3/2} = \frac{2}{3}\alpha_n^{3/2} e^{\mp 3i\pi/2}\left\{1 + \frac{3}{2}\frac{s^{2/3}y}{\alpha_n e^{\mp i\pi}} + O\left(\frac{1}{\alpha_n^2}\right)\right\} ,$$

where the upper sign applies when $\arg s > 0$, the lower one when $\arg s < 0$. Thus

$$\left|\mathrm{Ai}(s^{2/3}y - \alpha_n)\right| \sim \frac{1}{2\pi^{1/2}\alpha_n^{1/4}} \exp\left(\mp i\alpha_n^{1/2} y\, s^{2/3}\right) .$$

Hence the general term of (II.3) behaves, as $n \to \infty$, like

$$\frac{1}{8\pi\,\alpha_n \varepsilon_n} \exp\left\{-\alpha_n^{1/2} s^{2/3}\left[|x - x_o| \mp i(y + y_o)\right]\right\} .$$

Clearly, a necessary and sufficient condition for convergence is

$$\mathrm{Re}\left\{s^{2/3}\left[|x - x_o| \mp i(y + y_o)\right]\right\} > 0 ,$$

where the upper sign is taken for $\arg s > 0$, the lower one for $\arg s < 0$. Hence the series converges if

$$y + y_o < |x - x_o| \tan\left\{\left|\frac{2}{3}\arg s\right|\right\} . \qquad \text{(II.9)}$$

Now if (II.3) is to represent the Laplace transform $\bar{\phi}$ in the whole right-hand half-plane $\mathrm{Re}(s) > 0$, then (II.9) must hold for $|\arg s| < \frac{1}{2}\pi$, whence

$$y + y_o < \frac{|x - x_o|}{\sqrt{3}} . \qquad \text{(II.10)}$$

Since also $y > 0$, the domains in which the series for $\bar{\phi}$ is valid for $\mathrm{Re}(s) > 0$ are bounded above by two lines throught the image point $(x_o, -y_o)$ inclined at

an angle of 30° to the positive and the negative x-axes respectively, and, below, by the x-axes.

The shadow is given by $\delta > 0$, where δ is defined by (8.3). In the present case, $f = 0$, $g = 1 + y$, this becomes

$$\delta = \left| \int_{x_o}^{x} dx' \right| - \int_{o}^{y} \frac{dy'}{\sqrt{y'}} - \int_{o}^{y_o} \frac{dy'}{\sqrt{y'}} > 0 \ ,$$

or

$$|x - x_o| > 2\left(\sqrt{y} + \sqrt{y_o}\right) \ . \tag{II.11}$$

The relation between the domains defined by (II.9) and (II.10) is illustrated in Figure 2. Figure 2a shows the case $y_o = 0$. Here the 'convergence domain' (II.10) contains the whole of the boundary $y = 0$ (except for the point $x = x_o$) and, rather surprisingly, part of the reflection region $|x - x_o| > 2\sqrt{y}$ as well as a part of the shadow. Since every diffracted ray passes through the convergence domain, it may be possible in this case to extend the asymptotic formula for ϕ near the diffracted front by integration along a ray. Figure 2b shows a case where $y_o > (1 + \sqrt{2})/\sqrt{3}$. Then the lines $(y + y_o)\sqrt{3} = |x - x_o|$ do not meet the parabolic shadow boundary at all, and so the series expansions of $\bar{\phi}$ can apparently not be used in the complex inversion formula of the Laplace transform in a part of the shadow which contains some of the boundary $y = 0$.

Clearly, it is necessary to obtain an alternative representation of $\bar{\phi}$ which continues (II.3) analytically into the region

$$\frac{3}{2} \tan^{-1} \frac{y + y_o}{|x - x_o|} < |\arg s| < \frac{1}{2}\pi \ .$$

Formally, this is not difficult, as one can replace (II.3) by an integral which defines $\bar{\phi}$ for all s with Re(s) > 0. From this, an approximation to $\bar{\phi}$ can be obtained which is valid in the region of the xy-plane in which both the incident and the reflected pulses occur. Since this turns out to be the geometrical optics approximation, it can also be obtained directly. In general, there remains a region of the xy-plane (including at least a part of the shadow boundary) in which neither the geometrical optics approximation nor the approximation developed here is valid (or useful). The discussion of the behavior of $\bar{\phi}$ in this region is a problem which remains to be solved.

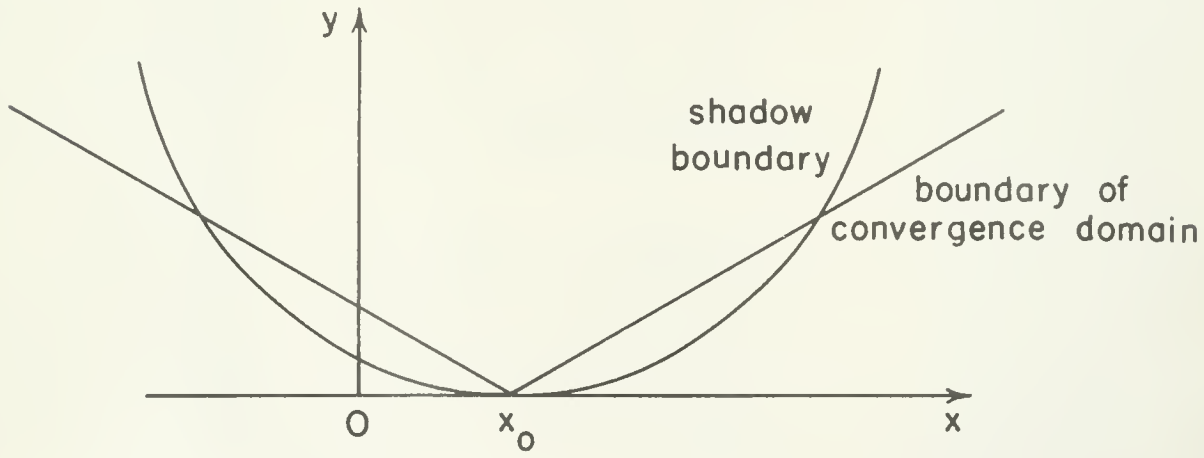

Figure 2a

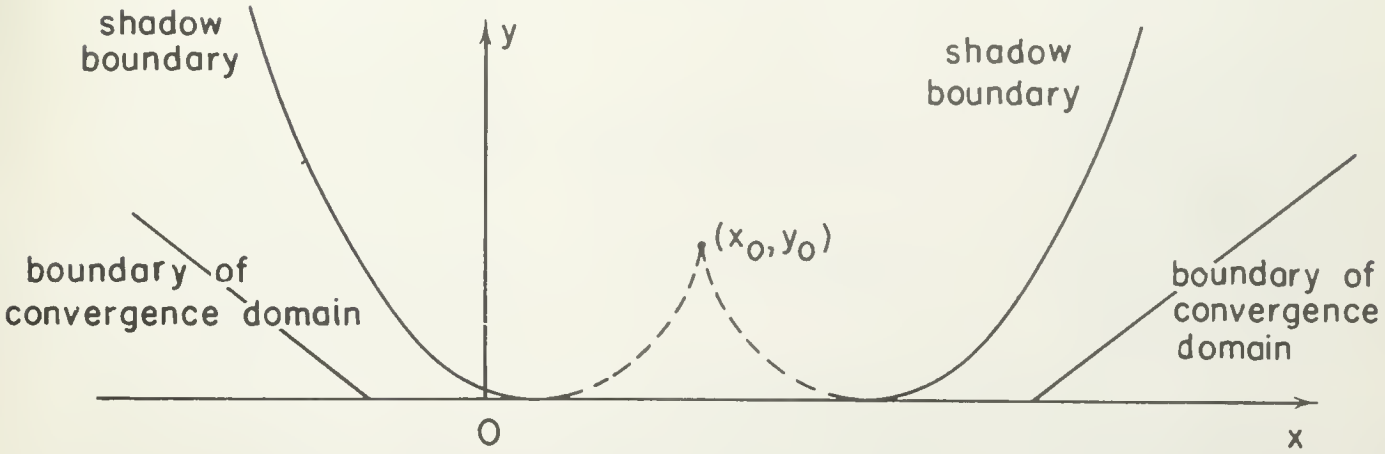

Figure 2b

References

[1] Friedlander, F.G. - On the Solutions of the wave equation with discontinuous derivatives; Proc. Camb. Phil. Soc., 38, 378-382 (1943).

[2] - Notes on the geometrical optics of diffracted wave fronts; Proc. Camb. Phil. Soc., 45, 395-401 (1949).

[3] - Diffraction of pulses by a circular cylinder; New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-64, 1954. Also Comm. Pure Appl. Math., 7, 705-732 (1954).

[4] Kline, Morris - An asymptotic solution of linear second-order hyperbolic differential equations, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-48, 1952. Also J. Rational Mechanics, 3, 315 (1954).

[5] Langer, R.E. - On the asymptotic solutions of ordinary differential equations; Trans. Amer. Math. Soc., 33, 23-64 (1931).

[6] Miller, J.C.P. - The Airy integral; British Association Mathematical Tables: Volume B, Cambridge, 1946.

[7] Titchmarsh, E.C. - Eigenfunction Expansions; Oxford Press, Cambridge, 1946.

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